

Mixed Threefolds Isogenous to a Product

Christian Gleissner

Abstract

In this paper we study *threefolds isogenous to a product of mixed type* i.e. quotients of a product of three compact Riemann surfaces C_i of genus at least two by the action of a finite group G , which is free, but not diagonal. In particular, we are interested in the systematic construction and classification of these varieties. Our main result is the full classification of threefolds isogenous to a product of mixed type with $\chi(\mathcal{O}_X) = -1$ assuming that any automorphism in G , which restricts to the trivial element in $\text{Aut}(C_i)$ for some C_i , is the identity on the product. Since the *holomorphic Euler-Poincaré-characteristic* of a smooth threefold of general type with ample canonical class is always negative, these examples lie on the boundary, in the sense of *threefold geography*. To achieve our result we use techniques from computational group theory. Indeed, we develop a MAGMA algorithm to classify these threefolds for any given value of $\chi(\mathcal{O}_X)$.

Introduction

A central aspect of algebraic geometry is to study symmetries of algebraic varieties i.e. their automorphisms. The goal in mind is to produce new and interesting varieties as quotients of known and well understood ones by groups of automorphisms. This idea was already applied very successfully by the classical geometers such as Godeaux et al. (see [G31]). The quotients that we are interested in were introduced by Catanese in [Cat00]:

A *variety X isogenous to a product of curves* is a quotient of a product of smooth complex projective curves C_i of genus at least two by the free action of a finite group of automorphisms:

$$X = (C_1 \times \dots \times C_n)/G.$$

The freeness of the action implies that X is a smooth projective variety of general type with ample canonical class K_X . In the last ten years several authors studied different aspects of the two-dimensional case extensively. In particular, surfaces isogenous to a product with *holomorphic Euler-Poincaré-characteristic* $\chi(\mathcal{O}_X) = 1$ were completely classified (see [BCG08, CP09, Pe10] et al). However, a systematic treatment of the higher-dimensional case was still missing until the author, in collaboration with Davide Frapporti [FG16], started to study these varieties in dimension three under the assumption that the action is unmixed i.e. each automorphism acts diagonally:

$$G \leq \text{Aut}(C_1) \times \text{Aut}(C_2) \times \text{Aut}(C_3).$$

Their main result is the full classification of these threefolds with $\chi(\mathcal{O}_X) = -1$ and a faithful G -action on each curve C_i . Geographically, these examples are extremal, since the holomorphic Euler-Poincaré characteristic of a smooth projective threefold of general type with ample canonical class is negative in contrast to the surface case where it is positive (cf. [Mi87]). In this paper we investigate the algebraically more involved *mixed case*, where G is acting non-diagonally. Our aim is to derive an algorithm, i.e. a finite procedure to classify mixed threefolds isogenous to a product for a fixed value of $\chi(\mathcal{O}_X)$. In particular, we want to determine the Galois groups G and the Hodge numbers $h^{p,q}(X)$ of these varieties. Similarly to the unmixed case, the technical condition that the induced actions $\psi_i: G_i \rightarrow \text{Aut}(C_i)$ of the groups

$$G_i := G \cap [\text{Aut}(C_1 \times \dots \times \widehat{C_i} \times \dots \times C_n) \times \text{Aut}(C_i)]$$

have trivial kernels K_i is imposed: we say that the G -action is *absolutely faithful*. This condition allows us to derive an effective bound for the order of the Galois group G , in terms of $\chi(\mathcal{O}_X)$, as well as numerical constraints on the genera of the curves C_i and the branching data T_i of the covers $C_i \rightarrow C_i/G_i$. These restrictions are very strong, indeed they allow only finitely many combinations, which are determined in the first step of our algorithm. Then, using Riemann's existence theorem, we classify all possible group actions $\psi_i: G_i \rightarrow \text{Aut}(C_i)$. In the next step, we determine all suitable groups G containing G_i that allow us to define a free and mixed G -action on the product $C_1 \times C_2 \times C_3$ via the maps ψ_i . Finally, we compute the *Hodge numbers* of the quotients

$$X = (C_1 \times C_2 \times C_3)/G.$$

This is achieved by analysing the induced representation of G on the *Dolbeault cohomology groups* $H^{p,q}(C_1 \times C_2 \times C_3)$. The classification procedure is computationally hard and cannot be carried out by hand, even in the boundary case $\chi(\mathcal{O}_X) = -1$. For this reason, the computer algebra system MAGMA [Mag] is used. We want to point out that the strategy of our algorithm differs slightly according to the index of the diagonal subgroup

$$G^0 := G \cap [\text{Aut}(C_1) \times \text{Aut}(C_2) \times \text{Aut}(C_3)]$$

in G . Since the quotient G/G^0 embeds naturally into the permutation group \mathfrak{S}_3 of the coordinates of the product (cf. Proposition 1.2), there are three cases:

$$G/G^0 \simeq \mathbb{Z}_2, \quad G/G^0 \simeq \mathfrak{A}_3 \quad \text{and} \quad G/G^0 \simeq \mathfrak{S}_3.$$

We call them index two, index three and index six case, respectively. In the index two case we can assume that the second and the third curve are isomorphic $C_2 \simeq C_3$, whereas all three curves C_i are isomorphic if the index is three or six.

Running our implementation for the boundary value $\chi(\mathcal{O}_X) = -1$ we obtain the following theorems.

Theorem 0.1. ¹ Let X be a threefold isogenous to a product of curves of mixed type. Assume that the action of G is absolutely faithful, $\chi(\mathcal{O}_X) = -1$ and the index of G^0 in G is two. Then, the tuple

$$[G, T_1, T_2, h^{3,0}(X), h^{2,0}(X), h^{1,0}(X), h^{1,1}(X), h^{1,2}(X), d]$$

appears in the table below. Conversely, each row in the table is realized by a family of threefolds, depending on d parameters, which is obtained by an absolutely faithful G -action.

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
1	\mathbb{Z}_2	$\langle 2, 1 \rangle$	$[2; -]$	$[2; -]$	5	7	4	18	24	6
2	\mathbb{Z}_4	$\langle 4, 1 \rangle$	$[0; 2^2, 4^2]$	$[2; -]$	4	4	2	11	16	4
3	\mathbb{Z}_4	$\langle 4, 1 \rangle$	$[2; -]$	$[0; 2^6]$	7	7	2	14	22	6
4	\mathbb{Z}_4	$\langle 4, 1 \rangle$	$[2; -]$	$[1; 2^2]$	5	6	3	14	20	5
5	\mathbb{Z}_2^2	$\langle 4, 2 \rangle$	$[2; -]$	$[0; 2^6]$	5	5	2	14	20	6
6	\mathbb{Z}_2^2	$\langle 4, 2 \rangle$	$[2; -]$	$[1; 2^2]$	4	5	3	14	19	5
7	\mathfrak{S}_3	$\langle 6, 1 \rangle$	$[2; -]$	$[0; 3^4]$	4	4	2	12	17	4
8	\mathbb{Z}_6	$\langle 6, 2 \rangle$	$[2; -]$	$[0; 3^4]$	5	5	2	12	18	4
9	\mathbb{Z}_8	$\langle 8, 1 \rangle$	$[2; -]$	$[0; 2^2, 4^2]$	5	5	2	12	18	4
10	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2 \rangle$	$[2; -]$	$[0; 2^2, 4^2]$	5	5	2	12	18	4
11	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2 \rangle$	$[0; 2^2, 4^2]$	$[1; 2^2]$	3	2	1	7	11	3
12	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2 \rangle$	$[2; -]$	$[0; 2^5]$	5	5	2	12	18	5
13	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$\langle 8, 2 \rangle$	$[2; -]$	$[0; 2^5]$	6	6	2	12	19	5
14	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[2; -]$	$[0; 2^2, 4^2]$	4	4	2	12	17	4
15	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[0; 2^2, 4^2]$	$[1; 2^2]$	3	2	1	8	12	3
16	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[1; 2]$	$[0; 2^6]$	3	2	1	8	12	4
17	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[1; 2]$	$[1; 2^2]$	2	2	2	8	11	3
18	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[2; -]$	$[0; 2^5]$	4	4	2	11	16	5
19	\mathcal{D}_4	$\langle 8, 3 \rangle$	$[2; -]$	$[0; 2^5]$	5	5	2	12	18	5
20	Q	$\langle 8, 4 \rangle$	$[2; -]$	$[0; 2^2, 4^2]$	6	6	2	12	19	4
21	\mathbb{Z}_2^3	$\langle 8, 5 \rangle$	$[2; -]$	$[0; 2^5]$	4	4	2	12	17	5
22	\mathcal{D}_5	$\langle 10, 1 \rangle$	$[2; -]$	$[0; 5^3]$	4	4	2	10	15	3
23	\mathbb{Z}_{10}	$\langle 10, 2 \rangle$	$[2; -]$	$[0; 5^3]$	4	4	2	12	17	3
24	$Dic12$	$\langle 12, 1 \rangle$	$[2; -]$	$[0; 2^2, 3^2]$	6	6	2	12	19	4
25	$Dic12$	$\langle 12, 1 \rangle$	$[2; -]$	$[0; 3, 6^2]$	5	5	2	10	16	3
26	\mathbb{Z}_{12}	$\langle 12, 2 \rangle$	$[2; -]$	$[0; 2^2, 3^2]$	5	5	2	12	18	4
27	\mathbb{Z}_{12}	$\langle 12, 2 \rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	12	17	3
28	\mathcal{D}_6	$\langle 12, 4 \rangle$	$[2; -]$	$[0; 2^2, 3^2]$	4	4	2	12	17	4
29	\mathcal{D}_6	$\langle 12, 4 \rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	10	15	3
30	\mathcal{D}_6	$\langle 12, 4 \rangle$	$[2; -]$	$[0; 2^2, 3^2]$	4	4	2	11	16	4
31	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$\langle 12, 5 \rangle$	$[2; -]$	$[0; 2^2, 3^2]$	5	5	2	12	18	4
32	$\mathbb{Z}_3 \times \mathbb{Z}_2^2$	$\langle 12, 5 \rangle$	$[2; -]$	$[0; 3, 6^2]$	4	4	2	12	17	3
33	\mathbb{Z}_{16}	$\langle 16, 1 \rangle$	$[2; -]$	$[0; 2, 8^2]$	4	4	2	12	17	3
34	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[1; 2]$	$[0; 2^2, 4^2]$	3	2	1	6	10	2
35	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	4	2	0	6	11	3
36	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	3	1	0	5	9	3
37	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	4	2	0	5	10	3
38	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	4	2	0	7	12	3
39	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[1; 2]$	$[0; 2^5]$	3	2	1	6	10	3
40	$\mathbb{Z}_2^2 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 3 \rangle$	$[1; 2]$	$[0; 2^5]$	4	3	1	6	11	3
41	$\mathbb{Z}_4 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 4 \rangle$	$[1; 2]$	$[0; 2^2, 4^2]$	3	2	1	6	10	2
42	$\mathbb{Z}_4 \rtimes_{\varphi} \mathbb{Z}_4$	$\langle 16, 4 \rangle$	$[1; 2]$	$[0; 2^2, 4^2]$	4	3	1	6	11	2
43	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$\langle 16, 5 \rangle$	$[2; -]$	$[0; 2, 8^2]$	4	4	2	12	17	3
44	M_{16}	$\langle 16, 6 \rangle$	$[2; -]$	$[0; 2, 8^2]$	5	5	2	10	16	3

¹We refer to Notation 0.3 for the definition of the groups in the table.

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
45	\mathcal{D}_8	$\langle 16, 7 \rangle$	$[2; -]$	$[0; 2, 8^2]$	4	4	2	10	15	3
46	\mathcal{D}_8	$\langle 16, 7 \rangle$	$[2; -]$	$[0; 2^3, 4]$	4	4	2	11	16	4
47	$SD16$	$\langle 16, 8 \rangle$	$[2; -]$	$[0; 2, 8^2]$	4	4	2	12	17	3
48	$SD16$	$\langle 16, 8 \rangle$	$[2; -]$	$[0; 4^3]$	4	4	2	11	16	3
49	$SD16$	$\langle 16, 8 \rangle$	$[0; 2, 4, 8]$	$[1; 2^2]$	3	2	1	7	11	2
50	$SD16$	$\langle 16, 8 \rangle$	$[2; -]$	$[0; 2^3, 4]$	5	5	2	11	17	4
51	$Dic16$	$\langle 16, 9 \rangle$	$[2; -]$	$[0; 2, 8^2]$	6	6	2	10	17	3
52	$Dic16$	$\langle 16, 9 \rangle$	$[2; -]$	$[0; 4^3]$	5	5	2	11	17	3
53	$\mathbb{Z}_2^2 \times \mathbb{Z}_4$	$\langle 16, 10 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	3	1	0	5	9	3
54	$\mathcal{D}_4 \times \mathbb{Z}_2$	$\langle 16, 11 \rangle$	$[2; -]$	$[0; 2^3, 4]$	4	4	2	11	16	4
55	$\mathcal{D}_4 \times \mathbb{Z}_2$	$\langle 16, 11 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^5]$	3	1	0	6	10	3
56	$Q \times \mathbb{Z}_2$	$\langle 16, 12 \rangle$	$[2; -]$	$[0; 4^3]$	5	5	2	11	17	3
57	$\mathcal{D}_4 *_\phi \mathbb{Z}_4$	$\langle 16, 13 \rangle$	$[2; -]$	$[0; 4^3]$	4	4	2	11	16	3
58	$\mathcal{D}_4 *_\phi \mathbb{Z}_4$	$\langle 16, 13 \rangle$	$[2; -]$	$[0; 2^3, 4]$	5	5	2	11	17	4
59	$Dic20$	$\langle 20, 1 \rangle$	$[2; -]$	$[0; 2, 5, 10]$	6	6	2	10	17	3
60	\mathbb{Z}_{20}	$\langle 20, 2 \rangle$	$[2; -]$	$[0; 2, 5, 10]$	4	4	2	12	17	3
61	\mathcal{D}_{10}	$\langle 20, 4 \rangle$	$[2; -]$	$[0; 2, 5, 10]$	4	4	2	10	15	3
62	$\mathbb{Z}_2^2 \times \mathbb{Z}_5$	$\langle 20, 5 \rangle$	$[2; -]$	$[0; 2, 5, 10]$	4	4	2	12	17	3
63	$Dic24$	$\langle 24, 4 \rangle$	$[2; -]$	$[0; 3, 4^2]$	5	5	2	11	17	3
64	$\mathfrak{S}_3 \times \mathbb{Z}_4$	$\langle 24, 5 \rangle$	$[2; -]$	$[0; 3, 4^2]$	4	4	2	11	16	3
65	$\mathfrak{S}_3 \times \mathbb{Z}_4$	$\langle 24, 5 \rangle$	$[2; -]$	$[0; 2^3, 3]$	5	5	2	11	17	4
66	\mathcal{D}_{12}	$\langle 24, 6 \rangle$	$[2; -]$	$[0; 2^3, 3]$	4	4	2	11	16	4
67	$Dic12 \times \mathbb{Z}_2$	$\langle 24, 7 \rangle$	$[2; -]$	$[0; 2, 6^2]$	5	5	2	10	16	3
68	$Dic12 \times \mathbb{Z}_2$	$\langle 24, 7 \rangle$	$[2; -]$	$[0; 2, 6^2]$	6	6	2	10	17	3
69	$Dic12 \times \mathbb{Z}_2$	$\langle 24, 7 \rangle$	$[2; -]$	$[0; 3, 4^2]$	5	5	2	11	17	3
70	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24, 8 \rangle$	$[2; -]$	$[0; 2^3, 3]$	5	5	2	11	17	4
71	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24, 8 \rangle$	$[2; -]$	$[0; 3, 4^2]$	4	4	2	11	16	3
72	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24, 8 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	10	15	3
73	$\mathbb{Z}_3 \rtimes_\varphi \mathcal{D}_4$	$\langle 24, 8 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	12	17	3
74	$\mathbb{Z}_6 \times \mathbb{Z}_4$	$\langle 24, 9 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	12	17	3
75	$\mathcal{D}_4 \times \mathbb{Z}_3$	$\langle 24, 10 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	11	16	3
76	$\mathcal{D}_4 \times \mathbb{Z}_3$	$\langle 24, 10 \rangle$	$[2; -]$	$[0; 2, 6^2]$	5	5	2	10	16	3
77	$\mathcal{D}_6 \times \mathbb{Z}_2$	$\langle 24, 14 \rangle$	$[2; -]$	$[0; 2^3, 3]$	4	4	2	11	16	4
78	$\mathcal{D}_6 \times \mathbb{Z}_2$	$\langle 24, 14 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	10	15	3
79	$\mathbb{Z}_2^2 \times \mathbb{Z}_3$	$\langle 24, 15 \rangle$	$[2; -]$	$[0; 2, 6^2]$	4	4	2	12	17	3
80	$\mathbb{Z}_2^2 \rtimes_\varphi \mathbb{Z}_8$	$\langle 32, 5 \rangle$	$[1; 2]$	$[0; 2, 8^2]$	2	1	1	6	9	1
81	$\mathbb{Z}_3^3 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32, 6 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^3, 4]$	3	1	0	4	8	2
82	$\mathbb{Z}_3^3 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32, 6 \rangle$	$[1; 2]$	$[0; 2^3, 4]$	3	2	1	6	10	2
83	$M16 \rtimes_\varphi \mathbb{Z}_2$	$\langle 32, 7 \rangle$	$[1; 2]$	$[0; 2^3, 4]$	3	2	1	6	10	2
84	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32, 9 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^3, 4]$	3	1	0	5	9	2
85	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32, 9 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^3, 4]$	3	1	0	4	8	2
86	$\mathcal{D}_4 \rtimes_\varphi \mathbb{Z}_4$	$\langle 32, 9 \rangle$	$[0; 2, 4, 8]$	$[0; 2^5]$	3	1	0	5	9	2
87	$\mathbb{Z}_4 \rtimes_\varphi \mathbb{Z}_8$	$\langle 32, 12 \rangle$	$[1; 2]$	$[0; 2, 8^2]$	3	2	1	6	10	1
88	$\mathcal{D}_4 \times \mathbb{Z}_4$	$\langle 32, 25 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^3, 4]$	3	1	0	4	8	2
89	$\mathbb{Z}_4 \rtimes_\varphi \mathcal{D}_4$	$\langle 32, 28 \rangle$	$[0; 2^2, 4^2]$	$[0; 2^3, 4]$	3	1	0	5	9	2
90	$SD16 \times \mathbb{Z}_2$	$\langle 32, 40 \rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	11	16	3
91	$\mathcal{D}_8 *_\phi \mathbb{Z}_4$	$\langle 32, 42 \rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	11	16	3
92	$\text{Hol}(\mathbb{Z}_8)$	$\langle 32, 43 \rangle$	$[2; -]$	$[0; 2, 4, 8]$	4	4	2	10	15	3
93	$SD16 \rtimes_\varphi \mathbb{Z}_2$	$\langle 32, 44 \rangle$	$[2; -]$	$[0; 2, 4, 8]$	5	5	2	10	16	3
94	$2O$	$\langle 48, 28 \rangle$	$[2; -]$	$[0; 3^2, 4]$	5	5	2	11	17	3
95	$\text{GL}(2, \mathbb{F}_3)$	$\langle 48, 29 \rangle$	$[2; -]$	$[0; 3^2, 4]$	4	4	2	11	16	3
96	$\text{SL}(2, 3) \times \mathbb{Z}_2$	$\langle 48, 32 \rangle$	$[2; -]$	$[0; 3^2, 4]$	5	5	2	11	17	3
97	$\text{SL}(2, 3) \rtimes_\varphi \mathbb{Z}_2$	$\langle 48, 33 \rangle$	$[2; -]$	$[0; 3^2, 4]$	4	4	2	11	16	3
98	$Dic24 \rtimes_\varphi \mathbb{Z}_2$	$\langle 48, 37 \rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	11	16	3
99	$\mathcal{D}_4 \times \mathfrak{S}_3$	$\langle 48, 38 \rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	10	15	3
100	$\mathcal{D}_4 \rtimes_\varphi \mathfrak{S}_3$	$\langle 48, 39 \rangle$	$[2; -]$	$[0; 2, 4, 6]$	5	5	2	10	16	3
101	$\mathbb{Z}_6 \rtimes_\varphi \mathcal{D}_4$	$\langle 48, 43 \rangle$	$[2; -]$	$[0; 2, 4, 6]$	4	4	2	11	16	3

No.	G	Id	T_1	T_2	$h^{3,0}$	$h^{2,0}$	$h^{1,0}$	$h^{1,1}$	$h^{1,2}$	d
102	$\mathfrak{S}_4 \times \mathbb{Z}_4$	$\langle 96, 186 \rangle$	$[0; 2^2, 4^2]$	$[0; 2, 4, 6]$	3	1	0	3	7	1
103	$\mathrm{GL}(2, \mathbb{F}_3) \times \mathbb{Z}_2$	$\langle 96, 189 \rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	11	16	3
104	$(Q \times \mathbb{Z}_2) \rtimes_{\varphi} \mathfrak{S}_3$	$\langle 96, 190 \rangle$	$[2; -]$	$[0; 2, 3, 8]$	5	5	2	10	16	3
105	$2O \rtimes_{\varphi} \mathbb{Z}_2$	$\langle 96, 192 \rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	11	16	3
106	$\mathrm{GL}(2, \mathbb{F}_3) \rtimes_{\varphi} \mathbb{Z}_2$	$\langle 96, 193 \rangle$	$[2; -]$	$[0; 2, 3, 8]$	4	4	2	10	15	3
107	$\mathrm{GL}(2, \mathbb{Z}_4)$	$\langle 96, 195 \rangle$	$[0; 2^2, 4^2]$	$[0; 2, 4, 6]$	3	1	0	4	8	1

Explanation. The table above is organized in the following way:

- The first column enumerates the examples.
- The second column reports the Galois group (see Notation 0.3 for the definition of the groups).
- The third column provides the MAGMA identifier of the Galois group: $\langle a, b \rangle$ denotes the b^{th} group of order a in the *Database of Small Groups* (see [Mag]).
- The types $T_i = [g'_i; m_{i,1}, \dots, m_{i,r_i}]$ in column 4 and 5 report the branching data of the covers $C_i \rightarrow C_i/G_i$. Here g'_i is the genus of C_i/G_i and the $m_{i,j}$'s are the branching orders. The types are written in a simplified way: e.g. $[0; 2, 2, 4, 4]$ is abbreviated by $[0; 2^2, 4^2]$.
- The remaining columns report the Hodge numbers $h^{p,q}(X)$ and the number d of parameters of the families (see Remark 3.6).

Theorem 0.2.

- a) In the index three case there exists a unique group G yielding a one-dimensional family of threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = -1$. The group $G \simeq \mathbb{Z}_9 \rtimes_{\varphi} \mathbb{Z}_3$ has MAGMA id $\langle 27, 4 \rangle$ and the Hodge numbers are:

$$h^{3,0}(X) = 4, \quad h^{2,0}(X) = 2, \quad h^{1,0}(X) = 0, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 10.$$

- b) There is no group acting absolutely faithful and freely on a product of curves such that the quotient X has $\chi(\mathcal{O}_X) = -1$ and the index of G^0 in G is six.

The above theorems complete the classification of threefolds isogenous to a product with absolutely faithful G -action and $\chi(\mathcal{O}_X) = -1$. For the full list of examples in the unmixed case we refer to [FG16, Theorem 0.1]. It is remarkable that there are no rigid examples in our classification neither in the mixed, nor in the unmixed case. In contrast, there are six examples of rigid surfaces S isogenous to a product with $\chi(\mathcal{O}_S) = 1$ (see [BCG08]); one of them was originally discovered by Beauville (see [Be83]). To obtain rigid, three-dimensional examples with $\chi(\mathcal{O}_X) = -1$, we need to allow non-trivial kernels K_i . Indeed, modifying Beauville's construction, we are able to give an example of such a threefold (see Example 5.9). It would be interesting to find more, or even try to classify them completely. In

addition, we also provide an example of a threefold X isogenous to a product, where the index of G^0 in G is six, the G -action is not absolutely faithful and $\chi(\mathcal{O}_X) = -1$ (see Example 5.8).

The paper is organized in the following way: in Section 1 we introduce varieties isogenous to a product and explain their basic properties. Section 2 is dedicated to the structure of mixed group actions on a product of three curves. In Section 3 we define the algebraic datum of a mixed threefold X isogenous to a product. Based on that, we show in Section 4 how to determine the Hodge numbers of a threefold X isogenous to a product from an algebraic datum of X . In Section 5 we develop an algorithm to classify threefolds isogenous to a product of mixed type for a fixed value of $\chi(\mathcal{O}_X)$, that are obtained by an absolutely faithful group action, and present our main result: the classification of these varieties in the case $\chi(\mathcal{O}_X) = -1$.

Notation 0.3. Throughout the paper all varieties are defined over the field of complex numbers and the standard notation from complex algebraic geometry is used, see for example [GH78]. Moreover, we have the following notations and definitions from group theory:

- The cyclic group of order n is denoted by \mathbb{Z}_n , the dihedral group of order $2n$ by \mathcal{D}_n and the symmetric and alternating group on n letters by \mathfrak{S}_n and \mathfrak{A}_n , respectively.
- The quaternion group of order 8 is defined as $Q := \langle -1, i, j, k \mid i^2 = j^2 = k^2 = ijk = -1 \rangle$.
- The groups $\mathrm{GL}(n, \mathbb{F}_q)$ and $\mathrm{SL}(n, \mathbb{F}_q)$ are the general linear and special linear groups of $n \times n$ matrices over the field \mathbb{F}_q .
- The holomorph $\mathrm{Hol}(G)$ of a group G is the semi-direct product $G \rtimes_{id} \mathrm{Aut}(G)$.
- Let G_1 and G_2 be groups with isomorphic subgroups $U_i \leq Z(G_i)$ and let $\phi: U_1 \rightarrow U_2$ be an isomorphism. The central product $G_1 *_\phi G_2$ is defined as the quotient of the direct product $G_1 \times G_2$ by the normal subgroup $\{(g_1, g_2) \in U_1 \times U_2 \mid \phi(g_1)g_2 = 1_{G_2}\}$.
- The dicyclic group of order $4n$ is $\mathrm{Dic}4n := \langle a, b, c \mid a^n = b^2 = c^2 = abc \rangle$.
- The semidihedral group of order 2^n is $\mathrm{SD}2^n := \langle a, b \mid a^{2^{(n-1)}} = b^2 = 1, bab = a^{2^{(n-2)}-1} \rangle$.
- The group M_{16} of order 16 is $M_{16} := \langle a, b \mid a^8 = b^2 = e, bab^{-1} = a^5 \rangle$.
- The binary octahedral group of order 48 is $2O := \langle a, b, c \mid a^4 = b^3 = c^2 = abc \rangle$.

Acknowledgements. The author thanks Ingrid Bauer, Fabrizio Catanese, Davide Frapporti, Roberto Pignatelli and Sascha Weigl for several suggestions and useful mathematical discussions.

1 Generalities

In this section we introduce the objects that we study and state some of their basic properties.

Definition 1.1. *A complex algebraic variety X isogenous to a product of curves is a quotient*

$$(C_1 \times \dots \times C_n)/G$$

of a product of compact Riemann surfaces C_i of genus at least two by a finite group acting freely on the product.

The freeness of the G -action implies that X is a projective manifold of general type with ample canonical class K_X . It also allows us to derive formulas for the Chern invariants $\chi(\mathcal{O}_X)$, $e(X)$ and K_X^n in terms of the group order and the genera of the Riemann surfaces (see [FG16, Proposition 1.2]):

$$\chi(\mathcal{O}_X) = \frac{(-1)^n}{|G|} \prod_{i=1}^n (g(C_i) - 1), \quad e(X) = 2^n \chi(\mathcal{O}_X) \quad \text{and} \quad K_X^n = (-1)^n n! 2^n \chi(\mathcal{O}_X).$$

To study group actions on a product of compact Riemann surfaces C_i of genus at least two, it is important to understand the structure of the automorphism group of the product. This group has a simple description in terms of the automorphism groups $\text{Aut}(C_i)$ of the factors:

Proposition 1.2 ([Cat00, Corollary 3.9]). *Let D_1, \dots, D_k be pairwise non isomorphic compact Riemann surfaces of genus at least two, then*

$$\text{Aut}(D_1^{n_1} \times \dots \times D_k^{n_k}) = (\text{Aut}(D_1)^{n_1} \rtimes \mathfrak{S}_{n_1}) \times \dots \times (\text{Aut}(D_k)^{n_k} \rtimes \mathfrak{S}_{n_k})$$

for all positive integers n_i .

Motivated by the proposition, we give the following definition:

Definition 1.3. *Let G be a subgroup of $\text{Aut}(C_1 \times \dots \times C_n)$, where $g(C_i) \geq 2$. Then we define*

$$G_i := G \cap (\text{Aut}(C_1 \times \dots \times \widehat{C_i} \times \dots \times C_n) \times \text{Aut}(C_i)) \quad \text{for all} \quad 1 \leq i \leq n.$$

Note that the elements of G_i are precisely those automorphisms that can be restricted to C_i . We write $\psi_i: G_i \rightarrow \text{Aut}(C_i)$ for the restriction homomorphism and denote its kernel by K_i . Clearly, the diagonal subgroup

$$G^0 := G \cap (\text{Aut}(C_1) \times \dots \times \text{Aut}(C_n)) \leq G$$

is equal to the intersection of the groups G_i and the quotient G/G^0 embeds naturally in the permutation group \mathfrak{S}_n of the factors of the product. If G/G^0 is trivial, we say that the action on the product is *unmixed* and otherwise *mixed*. Similarly, the quotient of the product by G is called of *unmixed* or *mixed type*, respectively. In this paper, unless otherwise stated, we will consider the mixed case (see [FG16] for the unmixed).

2 Mixed Actions

An unmixed action of a finite group G on a product of Riemann surfaces is given by

$$g(x_1, \dots, x_n) = (\psi_1(g)x_1, \dots, \psi_n(g)x_n), \quad \text{for all } g \in G.$$

In this section we show that after conjugation with a suitable automorphism in

$$\text{Aut}(C_1) \times \dots \times \text{Aut}(C_n)$$

there are analogous formulas describing a mixed G -action in terms of the maps ψ_i . Such a description, i.e. a *normalized form* of the action, is of great importance for the following reasons:

- it allows us to study the geometric properties of the quotient $(C_1 \times \dots \times C_3)/G$ using Riemann surface theory,
- the formulas defining the normal form can be used to construct an action of an abstract finite group G on a product of compact Riemann surfaces starting from suitable subgroups $G_i \leq G$ and group actions $\psi_i: G_i \rightarrow \text{Aut}(C_i)$.

For simplicity, we assume that $n = 3$, but similar results can be obtained in all dimensions. For $n = 2$ we refer the reader to [Cat00, Proposition 3.16 ii)]. According to the index of G^0 in G , there are three sub-cases of the mixed case:

$$G/G^0 \simeq \mathbb{Z}_2, \quad G/G^0 \simeq \mathfrak{A}_3 \quad \text{and} \quad G/G^0 \simeq \mathfrak{S}_3.$$

We call them index two, index three and index six case, respectively.

Convention: in the index two case we can assume that $C_2 \simeq C_3$. In the index three and six case it holds $C_1 \simeq C_2 \simeq C_3$. If we specialize in one of these cases, we may write $D \times C^2$ or C^3 instead of $C_1 \times C_2 \times C_3$.

Proposition 2.1. *Let G be a subgroup of the automorphism group of a product of three compact Riemann surfaces and $\nu: G \rightarrow G/G^0 \leq \mathfrak{S}_3$ be the projection map.*

- i) In the index two case we fix an element $\delta \in G$ of the form $\delta(x, y, z) = (\delta_1 x, \delta_3 z, \delta_2 y)$, i.e. $\nu(\delta) = (2, 3)$. Then, after conjugating with the automorphism $\xi(x, y, z) := (x, y, \delta_3 z)$, it holds*

$$\psi_3(g) = \psi_2(\delta g \delta^{-1}) \quad \text{for all } g \in G^0$$

and the action is given by the formulas

- $\delta(x, y, z) = (\psi_1(\delta)x, z, \psi_2(\delta^2)y)$
- $g(x, y, z) = (\psi_1(g)x, \psi_2(g)y, \psi_2(\delta g \delta^{-1})z) \quad \text{for all } g \in G^0.$

ii) In the index three case we fix an element $\tau \in G$ of the form $\tau(x, y, z) = (\tau_2 y, \tau_3 z, \tau_1 x)$, i.e. $\nu(\tau) = (1, 3, 2)$. Then, after conjugating with the automorphism $\epsilon(x, y, z) := (x, \tau_2 y, \tau_2 \tau_3 z)$, it holds

$$\psi_2(g) = \psi_1(\tau g \tau^{-1}) \quad \text{and} \quad \psi_3(g) = \psi_1(\tau^2 g \tau^{-2}) \quad \text{for all } g \in G^0$$

and the action is given by the formulas

- $\tau(x, y, z) = (y, z, \psi_1(\tau^3)x)$
- $g(x, y, z) = (\psi_1(g)x, \psi_1(\tau g \tau^{-1})y, \psi_1(\tau^2 g \tau^{-2})z)$ for all $g \in G^0$.

iii) In the index six case we fix an element $\tau \in G$ of the form $\tau(x, y, z) = (\tau_2 y, \tau_3 z, \tau_1 x)$, i.e. $\nu(\tau) = (1, 3, 2)$. Then, after conjugating with the automorphism $\epsilon(x, y, z) := (x, \tau_2 y, \tau_2 \tau_3 z)$, it holds

$$\psi_2(h) = \psi_1(\tau h \tau^{-1}) \quad \text{and} \quad \psi_3(k) = \psi_1(\tau^2 k \tau^{-2})$$

for all $h \in G_2$ and $k \in G_3$ and the action is given by the formulas

- $\tau(x, y, z) = (y, z, \psi_1(\tau^3)x)$
- $g(x, y, z) = (\psi_1(g)x, \psi_1(\tau g \tau^{-1})y, \psi_1(\tau^2 g \tau^{-2})z)$
- $f(x, y, z) = (\psi_1(f)x, \psi_1(\tau f \tau^{-2})z, \psi_1(\tau^2 f \tau^{-1})y)$

for all $g \in G^0$ and $f \in G_1 \setminus G^0$.

Since the proof of the Proposition is just a calculation, we skip it.

Convention: from now on we assume that a subgroup $G \leq \text{Aut}(C_1 \times C_2 \times C_3)$ is embedded in *normal form* for a fixed choice of δ or τ , respectively.

As already mentioned, a very important observation is that the formulas from Proposition 2.1 provide a way to define mixed group actions on a product of three compact Riemann surfaces:

Proposition 2.2. *Let G be a finite group with a normal subgroup G^0 such that G/G^0 is isomorphic to \mathbb{Z}_2 , \mathfrak{A}_3 or \mathfrak{S}_3 . Let $\nu: G \rightarrow G/G^0$ be the quotient map.*

i) In the index two case, let $\psi_1: G \rightarrow \text{Aut}(D)$ and $\psi_2: G^0 \rightarrow \text{Aut}(C)$ be group actions on compact Riemann surfaces with kernels K_i such that

$$K_1 \cap K_2 \cap \delta K_2 \delta^{-1} = \{1_G\}$$

for an element $\delta \in G \setminus G^0$. Then the formulas from Proposition 2.1 i) define an embedding $G \hookrightarrow \text{Aut}(D \times C^2)$.

ii) In the index three case, let $\alpha: G/G^0 \rightarrow \mathfrak{A}_3$ be an isomorphism and $\psi_1: G^0 \rightarrow \text{Aut}(C)$ be a group action on a compact Riemann surface with kernel K_1 such that

$$K_1 \cap \tau K_1 \tau^{-1} \cap \tau^2 K_1 \tau^{-2} = \{1_G\}$$

for an element $\tau \in G$ with $(\alpha \circ \nu)(\tau) = (1, 3, 2)$. Then the formulas from Proposition 2.1 ii) define an embedding $G \hookrightarrow \text{Aut}(C^3)$.

iii) In the index six case, let $\alpha: G/G^0 \rightarrow \mathfrak{S}_3$ be an isomorphism. Define

$$G_1 := (\alpha \circ \nu)^{-1}(\langle (2, 3) \rangle)$$

and let $\psi_1: G_1 \rightarrow \text{Aut}(C)$ be a group action on a compact Riemann surface with kernel K_1 such that

$$K_1 \cap \tau K_1 \tau^{-1} \cap \tau^2 K_1 \tau^{-2} = \{1_G\}$$

for an element $\tau \in G$ with $(\alpha \circ \nu)(\tau) = (1, 3, 2)$. Then the formulas from Proposition 2.1 iii) define an embedding $G \hookrightarrow \text{Aut}(C^3)$.

Definition 2.3 (cf. [FG16, Definition 3.1]). As in the unmixed case, we say that the G -action is minimal if $K_i \cap K_j$ is trivial for all $i \neq j$ and absolutely faithful if the kernels K_i are trivial.

We point out that every threefold isogenous to a product can be obtained by a unique minimal action (cf. [Cat00, Proposition 3.13]). Hence, from now on, we assume that the action of G is minimal. Note that the kernels K_i are related: we have $K_3 = \delta K_2 \delta^{-1}$ in the index two case, whereas $K_2 = \tau^2 K_1 \tau^{-2}$ and $K_3 = \tau K_1 \tau^{-1}$ in the index three and index six case.

In Proposition 2.2 we described the building data to define a mixed group action on a product of three curves. Since we want to construct threefolds isogenous to a product, we shall give conditions that ensure the freeness of the action on the product in terms of the maps ψ_i . Such conditions are easily deduced from the formulas in Proposition 2.1 (cf. [Cat00, Proposition 3.16 ii]) for the two dimensional case). To phrase them in a compact way, the following definition is convenient.

Definition 2.4. Let $\psi: H \rightarrow \text{Aut}(C)$ a group action on a Riemann surface. The stabilizer set $\Sigma \subset H$ of ψ is defined as the set of elements admitting at least one fixed point on C .

Proposition 2.5. Let G be a subgroup of the automorphism group of a product $C_1 \times C_2 \times C_3$ of compact Riemann surfaces and Σ_i be the stabilizer set of $\psi_i: G_i \rightarrow \text{Aut}(C_i)$. Then the freeness of the G -action is equivalent to:

a) index two case

i) $\Sigma_1 \cap \Sigma_2 \cap \delta \Sigma_2 \delta^{-1} = \{1_G\}$ and

ii) for all $g \in G^0$ with $\delta g \in \Sigma_1$, it holds $(\delta g)^2 \notin \Sigma_2$.

b) index three case

i) $\Sigma_1 \cap \tau \Sigma_1 \tau^{-1} \cap \tau^2 \Sigma_1 \tau^{-2} = \{1_G\}$ and

ii) for all $g \in G^0$ it holds $(\tau g)^3 \notin \Sigma_1$.

c) *index six case*

- i) $\Sigma_1 \cap \tau\Sigma_1\tau^{-1} \cap \tau^2\Sigma_1\tau^{-2} = \{1_G\}$,
- ii) *for all $g \in G^0$ it holds $(\tau g)^3 \notin \Sigma_1$ and*
- iii) *for all $f \in G_1 \setminus G^0$ with $f \in \Sigma_1$, it holds $\tau f^2\tau^{-1} \notin \Sigma_1$.*

Remark 2.6. For completeness, we want to mention that an unmixed action is free if and only the intersection $\Sigma_1 \cap \Sigma_2 \cap \Sigma_3$ is trivial.

Corollary 2.7. *Let G be a subgroup of the automorphism group of a product of three curves.*

- a) *Assume that $G^0 \trianglelefteq G$ is of index six and G is acting freely on the product, then the short exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{S}_3 \longrightarrow 1$$

does not split.

- b) *Assume that $G^0 \trianglelefteq G$ is of index three and condition i) in Proposition 2.5 b) holds. Then ii) in Proposition 2.5 b) is equivalent to the condition, that the short exact sequence*

$$1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{A}_3 \longrightarrow 1$$

does not split.

Proof. a) A short exact sequence $1 \longrightarrow G^0 \longrightarrow G \longrightarrow \mathfrak{S}_3 \longrightarrow 1$ splits, if and only if there exist elements $a, b \in G \setminus G^0$ such that $\text{ord}(a) = 2$, $\text{ord}(b) = 3$ and $aba = b^{-1}$. Assume that the sequence splits, then there exist elements a, b as above. Since $b^2 \notin G^0$ we can assume that $b = \tau g \in \tau G^0$. This leads to the contradiction $(\tau g)^3 = 1 \in \Sigma_1$. The proof of b) is similar. \square

3 The algebraic datum

The aim of this section is to give a *group theoretical description* of a threefold

$$X = (C_1 \times C_2 \times C_3)/G$$

isogenous to a product of mixed type. We refer the reader to [FG16, Section 3] for the unmixed case. In the previous section, we worked out a description of mixed actions using the maps

$$\psi_i: G_i \rightarrow \text{Aut}(C_i);$$

dividing by their kernels K_i , we obtain effective actions of the factor groups G_i/K_i on the curves C_i , which can be characterized by *Riemann's existence theorem*:

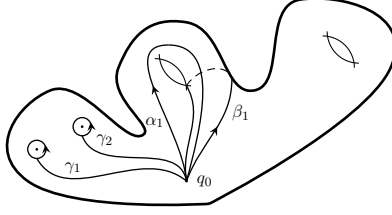


Figure 1: generators of π_1

Theorem 3.1 ([Mir95, cf. Sections III.3 and III. 4]). *An effective action $\psi: H \rightarrow \text{Aut}(C)$ of a finite group H on a compact Riemann surface C is given and completely described by the following*

- a compact Riemann surface C' ,
- a finite set $\mathcal{B} \subset C'$ (the branch points) and
- a surjective homomorphism $\eta: \pi_1(C' \setminus \mathcal{B}, q_0) \rightarrow H$ (the monodromy map).

Recall that the fundamental group of $C' \setminus \mathcal{B}$ has a presentation of the form

$$\pi_1(C' \setminus \mathcal{B}, q_0) = \langle \gamma_1, \dots, \gamma_r, \alpha_1, \beta_1, \dots, \alpha_{g'}, \beta_{g'} \mid \gamma_1 \cdots \gamma_r \cdot \prod_{i=1}^{g'} [\alpha_i, \beta_i] \rangle.$$

Here, the generators γ_i are simple loops around the branch points (see Figure 1).

Note that the images of the generators of $\pi_1(C' \setminus \mathcal{B}, q_0)$ under the monodromy map

$$h_i := \eta(\gamma_i), \quad a_i := \eta(\alpha_i) \quad \text{and} \quad b_i := \eta(\beta_i)$$

generate H and satisfy the relation

$$h_1 \cdots h_r \cdot \prod_{i=1}^{g'} [a_i, b_i] = 1 \quad (*).$$

This provides the motivation for our next definition.

Definition 3.2. *Let $m_1, \dots, m_r \geq 2$ and $g' \geq 0$ be integers and H be a finite group. A generating vector for H of type $[g'; m_1, \dots, m_r]$ is a $(2g' + r)$ -tuple*

$$(h_1, \dots, h_r, a_1, b_1, \dots, a_{g'}, b_{g'})$$

of group elements which generate H , satisfy the relation $()$ and fulfill the condition $\text{ord}(h_i) = m_i$ for all $1 \leq i \leq r$.*

Remark 3.3. Let $\psi: H \rightarrow \text{Aut}(C)$ be an effective group action and $V = (h_1, \dots, h_r, a_1, b_1, \dots, a_{g'}, b_{g'})$ be an associated generating vector of H . Since the cyclic groups $\langle h_i \rangle$ and their conjugates provide the

non-trivial stabilizers of the action, the stabilizer set of ψ can be written as

$$\Sigma = \bigcup_{h \in H} \bigcup_{i \in \mathbb{Z}} \bigcup_{j=1}^r \{hh_j^i h^{-1}\}.$$

For this reason it makes also sense to refer to Σ as the stabilizer set associated to the generating vector V .

Similarly to the unmixed case (see [FG16, Section 3]), we attach to a threefold isogenous to a product of mixed type certain algebraic data, reflecting the geometry of the threefold. We have the groups G and G^0 , the kernels K_i and the embedding $G/G^0 \leq \mathfrak{S}_3$. In the index three case we choose an element $\tau \in G$ with residue class $(1, 3, 2)$; in the index six case we choose elements $\tau, h \in G$ with classes $(1, 3, 2)$ and $(2, 3)$. For each $\overline{\psi_i}: G_i/K_i \rightarrow \text{Aut}(C_i)$ we can choose a generating vector V_i for G_i/K_i of type T_i . Note that the latter is not unique, only the type T_i is uniquely determined. Since we work in the mixed case, the actions ψ_i are related according to Proposition 2.1. This implies that some of the data is redundant. To keep track of it, we define:

Definition 3.4. *To a threefold X isogenous to a product of mixed type we attach the tuple*

- $(G, G^0, K_1, K_2, V_1, V_2)$ in the index two case,
- (G, G^0, K_1, τ, V_1) in the index three case,
- $(G, G^0, K_1, \tau, h, V_1)$ in the index six case,

and call it an algebraic datum of X .

Thanks to Riemann's existence theorem, Proposition 2.2 and 2.5 we have a way to construct threefolds isogenous to a product starting from group theoretical data. For the two dimensional analogue, we refer to [BCG08, Proposition 2.5].

Proposition 3.5. *Let G be a finite group and $G^0 \trianglelefteq G$ be a normal subgroup such that $G/G^0 \leq \mathfrak{S}_3$ and let $\nu: G \rightarrow G/G^0$ be the quotient map.*

a) *Assume that $G/G^0 \simeq \mathbb{Z}_2$. Let $\delta \in G \setminus G^0$, $K_1 \trianglelefteq G$ and $K_2 \trianglelefteq G^0$ be normal subgroups such that*

$$K_1 \cap K_2 = \{1_G\} \quad \text{and} \quad K_2 \cap \delta K_2 \delta^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G/K_1 and V_2 a generating vector for G^0/K_2 . Let $\Sigma_i \subset G$ be the pre-images of the stabilizer sets associated to the generating vectors V_i under the quotient maps

$$G \rightarrow G/K_1 \quad \text{and} \quad G^0 \rightarrow G^0/K_2.$$

Assume that the freeness conditions from Proposition 2.5 a) hold. Then there exists a threefold X isogenous to a product with algebraic datum

$$(G, G^0, K_1, K_2, V_1, V_2).$$

b) Assume that $G/G^0 \simeq \mathbb{Z}_3$. Let $\tau \in G \setminus G^0$ and $K_1 \trianglelefteq G^0$ such that

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G^0/K_1 and $\Sigma_1 \subset G^0$ be the pre-image of the stabilizer set associated to the generating vector V_1 under the quotient map

$$G^0 \rightarrow G^0/K_1.$$

Assume that the freeness conditions from Proposition 2.5 b) hold. Then there exists a threefold X isogenous to a product with algebraic datum

$$(G, G^0, K_1, \tau, V_1).$$

c) Assume that $G/G^0 \simeq \mathfrak{S}_3$. Let $\tau, h \in G \setminus G^0$ such that $\tau^2 \notin G^0$ and $h^2 \in G^0$. Define the subgroup

$$G_1 := \langle h, G^0 \rangle \leq G.$$

Let $K_1 \trianglelefteq G_1$ be a normal subgroup such that

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

Let V_1 be a generating vector for G_1/K_1 and $\Sigma_1 \subset G_1$ be the pre-image of the stabilizer set associated to the generating vector V_1 under the quotient map

$$G_1 \rightarrow G_1/K_1.$$

Assume that the freeness conditions from Proposition 2.5 c) hold. Then there exists a threefold X isogenous to a product with algebraic datum $(G, G^0, K_1, \tau, h, V_1)$.

Remark 3.6. In Proposition 3.5 we actually construct families of threefolds, they depend on the choice of the complex structure on the quotient curves C'_i and the branch points $\mathcal{B}_i \subset C'_i$, that is

- $3(g'_1 + g'_2) - 6 + r_1 + r_2$ parameters in the index two case and
- $3g'_1 - 3 + r_1$ parameters in the index three and index six case, respectively.

The integers $g'_i = g(C'_i)$ and $r_i = |\mathcal{B}_i|$ are given in terms of the types $T_i = [g'_i; m_{i,1}, \dots, m_{i,r_i}]$ of the generating vectors V_i .

4 The Hodge diamond

In this section we explain how to compute the *Hodge numbers* of a threefold $X = (C_1 \times C_2 \times C_3)/G$ isogenous to a product of mixed type from an algebraic datum of X .

The idea is to use representation theory: the action of G on the product induces representations

$$\phi_{p,q}: G \rightarrow \mathrm{GL} \left(H^{p,q}(C_1 \times C_2 \times C_3) \right), \quad g \mapsto [\omega \mapsto (g^{-1})^* \omega]$$

on the *Dolbeault cohomology groups*, whose characters are denoted by $\chi_{p,q}$. Since the action is free, the Hodge numbers $h^{p,q}(X)$ are given as the dimensions of the G -invariant parts of the Dolbeault groups $H^{p,q}(C_1 \times C_2 \times C_3)$ i.e.

$$h^{p,q}(X) = \frac{1}{|G|} \sum_{g \in G} \chi_{p,q}(g).$$

In the same way the maps $\psi_i: G_i \rightarrow \mathrm{Aut}(C_i)$ induce representations $\varphi_i: G_i \rightarrow \mathrm{GL} \left(H^{1,0}(C_i) \right)$. Their characters χ_{φ_i} can be easily determined from the generating vectors V_i in the algebraic datum of X , thanks to the formula of *Chevalley-Weil*: see [CW34] for the original version and [FG16, Section 2] for the relevant details in our situation. What remains to do is to determine the characters $\chi_{p,q}$ in terms of the characters χ_{φ_i} . For this task we need the description of the mixed action from Proposition 2.1 and *Künneth's formula for Dolbeault cohomology*:

Proposition 4.1 ([GH78, p.103-104]). *There is an isomorphism*

$$H^{p,q}(C_1 \times C_2 \times C_3) \simeq \bigoplus_{\substack{s_1+s_2+s_3=p \\ t_1+t_2+t_3=q}} H^{s_1,t_1}(C_1) \otimes H^{s_2,t_2}(C_2) \otimes H^{s_3,t_3}(C_3),$$

induced by the natural projections $p_i: C_1 \times C_2 \times C_3 \rightarrow C_i$.

Formulas for the restrictions of the characters $\chi_{p,q}$ to the diagonal subgroup G^0 are easily derived using the fact that the character of a direct sum of representations is the sum of the characters and the character of a tensor product is equal to the product of the characters.

Theorem 4.2 ([FG16, Theorem 3.7]). *For the restrictions of the characters $\chi_{p,q}$ it holds:*

- i) $\mathrm{Res}_{G^0}^G (\chi_{1,0}) = \chi_{\varphi_1} + \chi_{\varphi_2} + \chi_{\varphi_3},$
- ii) $\mathrm{Res}_{G^0}^G (\chi_{1,1}) = 2\Re(\chi_{\varphi_1} \overline{\chi_{\varphi_2}} + \chi_{\varphi_1} \overline{\chi_{\varphi_3}} + \chi_{\varphi_2} \overline{\chi_{\varphi_3}}) + 3\chi_{triv},$
- iii) $\mathrm{Res}_{G^0}^G (\chi_{2,0}) = \chi_{\varphi_1} \chi_{\varphi_2} + \chi_{\varphi_1} \chi_{\varphi_3} + \chi_{\varphi_2} \chi_{\varphi_3},$
- iv) $\mathrm{Res}_{G^0}^G (\chi_{2,1}) = \overline{\chi_{\varphi_1}} \chi_{\varphi_2} \chi_{\varphi_3} + \chi_{\varphi_1} \overline{\chi_{\varphi_2}} \chi_{\varphi_3} + \chi_{\varphi_1} \chi_{\varphi_2} \overline{\chi_{\varphi_3}} + 2(\chi_{\varphi_1} + \chi_{\varphi_2} + \chi_{\varphi_3}),$
- v) $\mathrm{Res}_{G^0}^G (\chi_{3,0}) = \chi_{\varphi_1} \chi_{\varphi_2} \chi_{\varphi_3}.$

Here, χ_{triv} denotes the trivial character.

Remark 4.3. Since the representations χ_{φ_i} are defined in terms of the actions ψ_i , they are related to each other in the same way as these actions (see Proposition 2.1) e.g. we have

$$\chi_{\varphi_2}(h) = \chi_{\varphi_1}(\tau h \tau^{-1}) \quad \text{and} \quad \chi_{\varphi_3}(k) = \chi_{\varphi_1}(\tau^2 k \tau^{-2})$$

for all $h \in G_2$ and $k \in G_3$ in the index six case and similar formulas in the other cases.

It remains to determine the values of the characters $\chi_{p,q}$ for elements outside of G^0 . Here we need a simple lemma from linear algebra:

Lemma 4.4. *Let A, B and C be endomorphisms of a finite dimensional vector space V , then:*

- i) the trace of the endomorphism of $V^{\otimes 2}$ given by $u \otimes v \mapsto Av \otimes Bu$ is the trace of $A \circ B$,*
- ii) the trace of the endomorphism of $V^{\otimes 3}$ given by $u \otimes v \otimes w \mapsto Av \otimes Bw \otimes Cu$ is the trace of $A \circ B \circ C$.*

Theorem 4.5. *The values of the characters $\chi_{p,q}$ for the elements outside of G^0 are displayed in the table below:*

	(1, 0)	(1, 1)	(2, 0)	(2, 1)	(3, 0)
$\chi_{p,q}(\delta g)$	$\chi_{\varphi_1}(\delta g)$	1	$-\chi_{\varphi_2}((\delta g)^2)$	$-\overline{\chi_{\varphi_1}(\delta g)}\chi_{\varphi_2}((\delta g)^2)$	$-\chi_{\varphi_1}(\delta g)\chi_{\varphi_2}((\delta g)^2)$
$\chi_{p,q}(\tau g)$	0	0	0	0	$\chi_{\varphi_1}((\tau g)^3)$
$\chi_{p,q}(\tau^2 g)$	0	0	0	0	$\chi_{\varphi_1}((\tau^2 g)^3)$
$\chi_{p,q}(f)$	$\chi_{\varphi_1}(f)$	1	$-\chi_{\varphi_2}(f^2)$	$-\overline{\chi_{\varphi_1}(f)}\chi_{\varphi_2}(f^2)$	$-\chi_{\varphi_1}(f)\chi_{\varphi_2}(f^2)$

- *the first row holds for all $\delta g \in \delta G^0$ in the index two case,*
- *the second and third row holds for all $\tau g \in \tau G^0$ and $\tau^2 g \in \tau^2 G^0$ in the index three as well as the index six case and*
- *the last row holds for all $f \in G_1 \setminus G^0$ in the index six case.*

Remark 4.6. Note that the table above gives the values of the characters $\chi_{p,q}$ for all elements in $G \setminus G^0$. In the index two and index three case this is clear. In the index six case it follows from the identities

$$G_1 \setminus G^0 = \tau(G_2 \setminus G^0)\tau^{-1} = \tau^2(G_3 \setminus G^0)\tau^{-2}$$

and the fact that a character is constant under conjugation.

proof of Theorem 4.5. Under the natural homomorphism $\nu: G \rightarrow G/G^0 \leq \mathfrak{S}_3$ an element in $G \setminus G^0$ maps to a three cycle or to a transposition. For this reason we will prove the theorem just in two cases:

- a) for $\tau g \in \tau G^0$ i.e. $\nu(\tau g) = (1, 3, 2)$ and b) for $\delta g \in \delta G^0$ i.e. $\nu(\delta g) = (2, 3)$.

For the elements contained in $\tau^2 G^0$ the computation is identical to a) and for $f \in G_1 \setminus G^0$ it is identical to b).

a) The inverse of an element $\tau g \in \tau G^0$ acts on C^3 via

$$(\tau g)^{-1}(x, y, z) = (\psi_1(g^{-1}\tau^{-3})z, \psi_1(\tau g^{-1}\tau^{-1})x, \psi_1(\tau^2 g^{-1}\tau^{-2})y).$$

Let $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ be a pure tensor in $H^{s_1, t_1}(C) \otimes H^{s_2, t_2}(C) \otimes H^{s_3, t_3}(C)$, where

$$s_1 + s_2 + s_3 = p \quad \text{and} \quad t_1 + t_2 + t_3 = q.$$

Under Künneth's isomorphism ω is mapped to $p_1^* \omega_1 \wedge p_2^* \omega_2 \wedge p_3^* \omega_3$. The pullback of this element via $(\tau g)^{-1}$ is:

$$\pm p_1^* \psi_1(\tau g^{-1}\tau^{-1})^* \omega_2 \wedge p_2^* \psi_1(\tau^2 g^{-1}\tau^{-2})^* \omega_3 \wedge p_3^* \psi_1(g^{-1}\tau^{-3})^* \omega_1,$$

where the sign depends on the degrees of the classes ω_i . The corresponding tensor

$$\pm \psi_1(\tau g^{-1}\tau^{-1})^* \omega_2 \otimes \psi_1(\tau^2 g^{-1}\tau^{-2})^* \omega_3 \otimes \psi_1(g^{-1}\tau^{-3})^* \omega_1$$

is an element in

$$H^{s_2, t_2}(C) \otimes H^{s_3, t_3}(C) \otimes H^{s_1, t_1}(C).$$

We conclude that ω and $((\tau g)^{-1})^* \omega$ are contained in different direct summands for all pairs

$$(p, q) \in \{(1, 0), (1, 1), (2, 0), (2, 1)\}.$$

This implies that the traces of the linear maps

$$((\tau g)^{-1})^*: H^{p, q}(C^3) \rightarrow H^{p, q}(C^3)$$

are equal to zero for these pairs. In other words $\chi_{p, q}(\tau g) = 0$. In the case $(p, q) = (3, 0)$ the forms ω_i are all of type $(1, 0)$. Therefore, the sign in the formula for the pullback of ω is $+1$ and there is only one summand in the decomposition of $H^{3, 0}(C^3)$. It holds

$$((\tau g)^{-1})^* \omega = \varphi_1(\tau g \tau^{-1}) \omega_2 \otimes \varphi_1(\tau^2 g \tau^{-2})^* \omega_3 \otimes \varphi_1(\tau^3 g)^* \omega_1.$$

We apply Lemma 4.4 ii) setting $A := \varphi_1(\tau g \tau^{-1})$, $B := \varphi_1(\tau^2 g \tau^{-2})$ and $C := \varphi_1(\tau^3 g)$ and obtain

$$\chi_{3, 0}(\tau g) = \text{tr}(ABC) = \text{tr}(\varphi_1(\tau g)^3) = \chi_{\varphi_1}((\tau g)^3).$$

b) take an element $\delta g \in \delta G^0$ and a pure tensor $\omega = \omega_1 \otimes \omega_2 \otimes \omega_3$ in

$$H^{s_1, t_1}(D) \otimes H^{s_2, t_2}(C) \otimes H^{s_3, t_3}(C) \subset H^{p, q}(D \times C^2).$$

The pullback of ω via $(\delta g)^{-1}$ is

$$((\delta g)^{-1})^* \omega = \pm \psi_1(g^{-1}\delta^{-1})^* \omega_1 \otimes \psi_2(\delta g^{-1}\delta^{-1})^* \omega_3 \otimes \psi_2(g^{-1}\delta^{-2})^* \omega_2.$$

This is a tensor in $H^{s_1, t_1}(D) \otimes H^{s_3, t_3}(C) \otimes H^{s_2, t_2}(C)$. For all pairs (p, q) , there is exactly one direct summand of $H^{p, q}(D \times C^2)$ containing both ω and $((\delta g)^{-1})^* \omega$. This implies that the trace of $((\delta g)^{-1})^*$ is equal to the trace of the restriction of $((\delta g)^{-1})^*$ to this summand. Using Lemma 4.4 *i*) in the same way as above, we get

(p, q)	invariant summand	$\chi_{p, q}(\delta g)$
$(1, 0)$	$H^{1, 0}(D) \otimes H^{0, 0}(C) \otimes H^{0, 0}(C)$	$\chi_{\varphi_1}(\delta g)$
$(1, 1)$	$H^{1, 1}(D) \otimes H^{0, 0}(C) \otimes H^{0, 0}(C)$	1
$(2, 0)$	$H^{0, 0}(D) \otimes H^{1, 0}(C) \otimes H^{1, 0}(C)$	$-\chi_{\varphi_2}((\delta g)^2)$
$(2, 1)$	$H^{0, 1}(D) \otimes H^{1, 0}(C) \otimes H^{1, 0}(C)$	$-\overline{\chi_{\varphi_1}(\delta g)} \chi_{\varphi_2}((\delta g)^2)$
$(3, 0)$	$H^{1, 0}(D) \otimes H^{1, 0}(C) \otimes H^{1, 0}(C)$	$-\chi_{\varphi_1}(\delta g) \chi_{\varphi_2}((\delta g)^2)$

□

5 Combinatorics, Bounds and Algorithms

Given a threefold isogenous to a product $X = (C_1 \times C_2 \times C_3)/G$, we consider the following numerical information:

- the group order $n := |G|$,
- the orders $k_i := |K_i|$ of the kernels of the maps $\psi_i: G_i \rightarrow \text{Aut}(C_i)$ and
- the types $T_i = [g'_i; m_{i,1}, \dots, m_{i,r_i}]$ (see Section 3) of the corresponding Galois covers

$$C_i \rightarrow C_i/\overline{G_i}, \quad \text{where} \quad \overline{G_i} := G_i/K_i.$$

Note that the collection above determines the genera $g_i := g(C_i)$ via Hurwitz' formula

$$g_i = \frac{|G_i|}{2k_i} \left(2g'_i - 2 + \sum_{j=1}^{r_i} \frac{m_{i,j} - 1}{m_{i,j}} \right) + 1,$$

and therefore also the invariants $\chi(\mathcal{O}_X)$, $e(X)$ and K_X^3 of the threefold X (see Section 1). In analogy to the definition of an algebraic datum (see Definition 3.4) we define:

Definition 5.1. *The numerical datum of a threefold X isogenous to a product is the tuple*

- $\mathcal{D} := (n, k_1, k_2, T_1, T_2)$ in the index two case and
- $\mathcal{D} := (n, k_1, T_1)$ in the index three and index six case.

If the action is absolutely faithful $k_i = 1$ for all $1 \leq i \leq 3$. Here, as a convention, we omit writing the k'_i s. Clearly, an algebraic datum \mathcal{A} of X determines the numerical datum \mathcal{D} of X and we say that the numerical datum \mathcal{D} is realized by the algebraic datum \mathcal{A} . We point out that $k_2 = k_3$ and $T_2 = T_3$ in the index two case, whereas $k_1 = k_2 = k_3$ and $T_1 = T_2 = T_3$ in the index three and six case.

In this section we derive *combinatorial constraints* on the numerical data. These constraints will imply that there are only finitely many possibilities for the numerical data, once the value of $\chi(\mathcal{O}_X)$ is fixed. Consequently there can be only finitely many algebraic data realizing these numerical data. This fact can be turned into an algorithm searching systematically through all possibilities and thereby classifying all threefolds isogenous to a product with a fixed value of $\chi(\mathcal{O}_X)$.

Definition 5.2. We define the function $N_{\max}: \mathbb{N}_{\geq 2} \rightarrow \mathbb{N}$, where

$$N_{\max}(g) := \max \{ |\text{Aut}(C)| \mid C \text{ is a compact Riemann surface with } g(C) = g \}.$$

According to Hurwitz' classical result $N_{\max}(g)$ is bounded by $84(g-1)$. However, for many values of g , the quantity $N_{\max}(g)$ is actually much smaller. Conder's paper [Con14] contains a table that displays all $N_{\max}(g)$ in the range $2 \leq g \leq 301$. It is the most comprehensive reference that we found and it will be very useful for our computations.

Proposition 5.3. Let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to a product of mixed type with numerical datum \mathcal{D} . Then

$$n \leq \left\lfloor \sqrt{-d \cdot \chi(\mathcal{O}_X) \prod_{i=1}^3 \frac{k_i}{\Theta_{\min}(T_i)}} \right\rfloor \quad \text{in the general case and} \quad n \leq \lfloor 42\sqrt{-d \cdot 42\chi(\mathcal{O}_X)} \rfloor$$

if the action is absolutely faithful. Here

$$\Theta_{\min}(T_i) := \begin{cases} 1/42, & \text{if } g'_i = 0 \\ 1/2, & \text{if } g'_i = 1, \\ 2g'_i - 2, & \text{if } g'_i \geq 2 \end{cases} \quad \text{for } T_i = [g'_i; m_{i,1}, \dots, m_{i,r_i}]$$

and the parameter d is defined as 32 in the index two case and as 216 in the index three and index six case.

For a proof we refer to [FG16, Proposition 4.4 and Corollary 4.5], which are the analogous statements in the unmixed case; indeed setting $d = 8$, the formulas in the proposition become the bounds in the unmixed case.

Note that in the absolutely faithful case, the bound for $n = |G|$ is solely in terms of $\chi(\mathcal{O}_X)$. Such a result can also be derived in the general case: let $X = (C_1 \times C_2 \times C_3)/G$ be a threefold isogenous to

a product of mixed type, then X is covered by an unmixed threefold

$$X^0 := (C_1 \times C_2 \times C_3)/G^0 \quad \text{and it holds} \quad \chi(\mathcal{O}_{X^0}) = |G/G^0| \chi(\mathcal{O}_X).$$

According to [FG16, Proposition 4.6]:

$$|G^0| \leq 84^6 \chi(\mathcal{O}_{X^0})^2 \quad \text{which yields the bound} \quad n = |G| \leq 84^6 |G/G^0|^3 \chi(\mathcal{O}_X)^2.$$

Unfortunately, even in the simplest case, when the holomorphic Euler-Poincaré-characteristic is -1 this bound is too large to be useful from the computational point of view. It would be interesting to understand if there exists a significantly better bound for $n = |G|$ in terms of $\chi(\mathcal{O}_X)$.

Proposition 5.4. *Let X be a threefold isogenous to a product, with algebraic datum \mathcal{D} . Then*

- i) $k_i \mid (g_{[i+1]} - 1)(g_{[i+2]} - 1)$,
- ii) $m_{i,j} \mid (g_{[i+1]} - 1)(g_{[i+2]} - 1)$,
- iii) $(g_i - 1) \mid \chi(\mathcal{O}_X) \frac{n}{k_i}$,
- iv) $r_i \leq \frac{4d_i k_i (g_i - 1)}{n} - 4g'_i + 4$,
- v) $m_{i,j} \leq 4g_i + 2$,
- vi) $g'_i \leq 1 - \frac{d_i k_i \chi(\mathcal{O}_X)}{(g_{[i+1]} - 1)(g_{[i+2]} - 1)} \leq 1 - d_i \chi(\mathcal{O}_X)$.
- vii) $n/(k_i d_i) \leq N_{\max}(g_i)$

Here, $[\cdot]$ denotes the residue mod 3 and

- $d_1 = 1$ and $d_2 = d_3 = 2$ in the index two case,
- $d_i = 3$ for all i in the index three and index six case.

Also here we obtain the analogous constraints from unmixed case setting $d_i = 1$ for all i (cf. [FG16, Proposition 4.8]). As an immediate consequence of Proposition 5.3 and Proposition 5.4 we conclude that there are only finitely many algebraic data of threefolds X isogenous to a product with

$$\chi(\mathcal{O}_X) \leq -1.$$

For completeness, we want to state the following useful, but trivial Remark:

Remark 5.5.

- a) In the index two case $g_2 = \sqrt{\frac{-n \cdot \chi(\mathcal{O}_X)}{g_1 - 1}} + 1$.
- b) In the index three and index six case $g_1 = \sqrt[3]{-n \cdot \chi(\mathcal{O}_X)} + 1$.

The combinatorial constraints that we found enable us to give an algorithm to classify threefolds isogenous to a product with a fixed value of $\chi(\mathcal{O}_X)$. Since the bound for the group order is very large in the general case, a complete classification, even with the help of a computer and just for small values of $\chi(\mathcal{O}_X)$, seems to be out of reach. On the other hand, if the group action is assumed to be absolutely faithful, then the bound drops significantly and a full classification, at least for $\chi(\mathcal{O}_X) = -1$, is possible. For this reason, we restrict ourselves to the absolutely faithful case. The exact strategy that we follow in our algorithm differs slightly according to the index of G^0 in G . Our MAGMA implementation is based on the code given in [BCGP12, Appendix]. We point out that the program relies heavily on *MAGMA's Database of Small Groups* (see [Mag]), which contains:

- all groups of order up to 2000, excluding the groups of order 1024,
- the groups whose order is a product of at most 3 primes,
- the groups of order dividing p^6 for p prime,
- the groups of order $p^k q$, where p^k is a prime-power dividing 2^8 , 3^6 , 5^5 or 7^4 and q is a prime different from p .

Since the full code is very long, we just explain the strategy.²

Input: A value χ for the holomorphic Euler-Poincaré-characteristic.

Part 1: In the first part we determine the set of *admissible numerical data*. This is the finite set of tuples of the form

- (n, T_1, T_2) in the index two case and
- (n, T_1) in the index three and index six case,

such that the combinatorial constraints from Proposition 5.4 and Remark 5.5, the inequality from Proposition 5.3 and Hurwitz' formula are satisfied.

Note that the set of numerical data of threefolds isogenous to a product with $\chi(\mathcal{O}_X) = \chi$ is a subset of the set of admissible numerical data.

In our implementation, this computation is performed by the functions `AdNDindexTwo`, `AdNDindexThree` and `AdNDindexSix` in the respective cases. The functions just return the set of admissible numerical data such that the groups of order n in the unmixed case, $n/2$ in the index two case and $n/3$ in the index three and index six case are contained in the Database of Small Groups. The exceptions are stored in the files `ExcepIndexTwo χ .txt`, `ExcepIndexThree χ .txt` and `ExcepIndexSix χ .txt`.

Part 2: In the second part of the algorithm, we search for algebraic data.

Index two case:

²The interested reader can find the full code at <http://www.staff.uni-bayreuth.de/~bt300503/>.

Step 1: Starting from the triples (n, T_1, T_2) contained in the set $\text{AdNDIndexTwo}(\chi)$, compute the set of 4-tuples (n, T_1, T_2, H) , where H is a group of order $n/2$ admitting at least one generating vector of type T_2 .

In our implementation, this computation is performed by the function `NDHIndexTwo`. The set of 4-tuples (n, T_1, T_2, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining tuples are stored in the file `ExcepIndexTwo χ .txt`.

Step 2: For each integer n belonging to some 4-tuple in the set $\text{NDHIndexTwo}(\chi)$ consider the groups of order n . For each group G of order n construct the list of subgroups of index two. For each G^0 in this list consider the 4-tuples (n, T_1, T_2, H) from Step 1 such that $H \simeq G^0$. For each of this 4-tuples compute the set of generating vectors V_1 for G of type T_1 and the set of generating vectors V_2 for G^0 of type T_2 . Check the freeness conditions $i)$ and $ii)$ of Proposition 2.5 $b)$. If they are fulfilled, then there exists a threefold X isogenous to a product with algebraic datum (G, G^0, V_1, V_2) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 3.5). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, T_2, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexTwo χ .txt`. Step 2 is performed calling `ClassifyIndexTwo χ` .

Index three case:

Step 1: Starting from the pairs (n, T_1) contained in the set $\text{AdNDIndexThree}(\chi)$, compute the set of triples (n, T_1, H) , where H is a group of order $n/3$ admitting three generating vectors V_1, V'_1 and V''_1 of type T_1 such that the associated stabilizer sets Σ_1, Σ'_1 and Σ''_1 fulfill the condition

$$\Sigma_1 \cap \Sigma'_1 \cap \Sigma''_1 = \{1_H\}.$$

Here we use the fact that a threefold isogenous to a product of mixed type with numerical datum (n, T_1) is covered by a threefold of unmixed type, where $|G^0| = n/3$.

In our implementation, this computation is performed by the function `NDHIndexThree`. The set of triples (n, T_1, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining triples are stored in the file `ExcepIndexThree χ .txt`.

Step 2: For each integer n belonging to a triple from Step 1 consider the groups of order n . For each group G of order n construct the list of normal subgroups G^0 of index three such that the short exact sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow \mathfrak{A}_3 \rightarrow 1$$

does not split. For each G^0 in this list consider the triples (n, T_1, H) from Step 1 such that $H \simeq G^0$. For each of these 4-tuples choose an element $\tau \in G \setminus G^0$ and compute all generating vectors V_1 for G^0 of type T_1 . Check the freeness condition $i)$ of Proposition 2.5 $c)$. If it holds, then the second condition of the proposition is also fulfilled, since the sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow \mathfrak{A}_3 \rightarrow 1$$

is non-split which is an equivalent condition according to Proposition 2.7. Therefore, there exists a threefold X isogenous to a product with algebraic datum (G, G^0, τ, V_1) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 3.5). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexThree χ .txt`. Step 2 is performed calling `ClassifyIndexThree(χ)`.

Index six case:

Step 1: Starting from the pairs (n, T_1) contained in the set `AdNDIndexSix(χ)`, compute the set of triples (n, T_1, H) , where H is a group of order $n/3$ admitting a generating vector V_1 of type T_1 .

In our implementation, this computation is performed by the function `NDHIndexSix`. The set of triples (n, T_1, H) such that the groups of order n are contained in the Database of Small Groups is returned. The remaining triples are stored in the file `ExcepIndexSix χ .txt`.

Step 2: For each integer n belonging to a triple from Step 1 consider the list of groups of order n . For each group G of order n , consider the triples of the form (n, T_1, H) such that G admits a subgroup of index three isomorphic to H . Compute the set of normal subgroups G^0 of G of index six such that the short exact sequence

$$1 \rightarrow G^0 \rightarrow G \rightarrow \mathfrak{S}_3 \rightarrow 1$$

does not split. Choose elements $\tau, h \in G \setminus G^0$ such that $\tau^2 \notin G^0$ and $h^2 \in G^0$. If the group $G_1 := G^0 \cup h \cdot G^0$ is isomorphic to H , then compute all generating vectors V_1 of type T_1 for this group. For each of these vectors compute the associated stabilizer set Σ_1 and check the freeness conditions *i*), *ii*) and *iii*) of Proposition 2.5 *d*). If they are fulfilled, then there exists a threefold X isogenous to a product with algebraic datum (G, G^0, τ, h, V_1) and $\chi(\mathcal{O}_X) = \chi$ (see Proposition 3.5). Compute the Hodge diamond of X and save the occurrence

$$[G, T_1, h^{3,0}, h^{2,0}, h^{1,0}, h^{1,1}, h^{1,2}]$$

in the file `IndexSix χ .txt`. Step 2 is performed calling `ClassifyIndexSix(χ)`.

Computational Remark 5.6.

- In Part 2 of the algorithm we search for generating vectors. We point out that different generating vectors may determine threefolds with the same invariants. For example, this happens if (but not only if) they differ by some *Hurwitz moves*. These moves are described in [CLP15], [Zim87] and [Pe10] and we refer to these sources for further details.
- We point out that for a generating vector of type $[g'; -]$ the stabilizer set is trivial and the corresponding character χ_φ is the sum of the trivial character and $(g' - 1)$ copies of the regular character according to the formula of Chevalley-Weil (see [CW34]). Consequently, in this case it is sufficient for us to know the existence of a generating vector, but there is no need to compute all of them.

Main Computation.

We execute the implementation for the input value $\chi = -1$. Note that the combinatorial constraints in Part 1 of the program are very strong, so relatively few admissible numerical data are returned. The total number of admissible group orders turns out to be relatively small and the maximum possible group order drops significantly compared to the theoretical bound from Proposition 5.3. The table below summarizes the occurrences:

	index two	index three	index six
No. AdNumData	253	8	5
No. G-Orders	39	2	1
n_{max}	576	216	216
n_{theo}	1539	4000	4000

In the first row we report the total number of admissible numerical data, in the second row the total number of group orders, in the third row the maximum possible group order after performing Part 1 of the algorithm and in the last row the theoretical bound for the group order according to Proposition 5.3. There are no exceptional numerical data to be considered, i.e. the files `ExcepIndexTwo χ .txt`, `ExcepIndexThree χ .txt` and `ExcepIndexSix χ .txt` remain empty. The table below reports the computation time to run the complete program (Part 1 and Part 2) on a $8 \times 2.5\text{GHz}$ Intel Xenon L5420 workstation with 16GB RAM in the respective cases:

	index two	index three	index six
time	10h 28min	24 sec	30 sec

This computation yields our main result: the classification of threefolds isogenous to a product of mixed type with $\chi(\mathcal{O}_X) = -1$ and absolutely faithful G -action (see Theorem 0.1 and Theorem 0.2).

Computational Remark 5.7. Running Part 1 of the program for values of χ different from -1 exceptional numerical data might occur. We tried and executed Part 1 in the index two, index three and index six case for all values of χ in the range

$$-40 \leq \chi \leq -1$$

and found no exceptional numerical data. Albeit it is not of great importance in our context, we shall mention that there are methods to deal with the exceptional numerical data, if they should occur for $\chi \leq -41$. We refer the reader to the paper [BCG08], where the authors classify surfaces isogenous to a product with $p_g = q = 0$ and the analogous problem appears. Their strategy can be easily adapted to the threefold case. Nevertheless, running Part 2 of the program for χ different from -1 is very time and memory consuming, in particular in the index two case: when we decrease χ , then the maximal possible value for g'_i increases, according to Proposition 5.4 vi). Similarly, the maximal length r_i of the types

$$T_i = [g'_i; m_{i,1}, \dots, m_{i,r_i}]$$

that we obtain increases as well. This leads to a large number of generating vectors to be analysed, which slows down the computation and requires a lot of memory.

To conclude this chapter we give two further examples of threefolds X isogenous to a product with $\chi(\mathcal{O}_X) = -1$. The first one is of mixed type, obtained by an index six action, the second one is of unmixed type without parameters, i.e. a rigid example. Note that we have no index six example and no rigid examples in the absolutely faithful case with $\chi(\mathcal{O}_X) = -1$ neither of mixed nor unmixed type (see [FG16, Theorem 0.1]). Therefore, to produce these examples, we have to allow non-trivial kernels.

Example 5.8.

We begin with the index six example. Consider the group $G := \text{SmallGroup}(216, 90)$, it admits a unique normal subgroup G^0 such that $G/G^0 \simeq \mathfrak{S}_3$. Moreover, the extension

$$1 \rightarrow G^0 \rightarrow G \rightarrow \mathfrak{S}_3 \rightarrow 1$$

is non-split. For the elements $h := G.1 * G.2 * G.4^2$ and $\tau := G.3 * G.4^2$ in $G \setminus G^0$ it holds

$$\tau^2 \notin G^0 \quad \text{and} \quad h^2 \in G^0,$$

i.e. h and τ define an isomorphism $G/G^0 \rightarrow \mathfrak{S}_3$. The cyclic group K_1 generated by $G.1.3 * G.1.4$ is the unique normal subgroup in $G_1 := \langle h, G^0 \rangle$ of order six such that

$$K_1 \cap \tau K_1 \tau^{-1} = \{1_G\}.$$

The quotient G_1/K_1 is isomorphic to the dihedral group \mathcal{D}_6 via the map $G_1/K_1 \rightarrow \mathcal{D}_6$ defined by

$$\overline{G.1.1} \mapsto s \quad \text{and} \quad \overline{G.1.2 * G.1.5} \mapsto t.$$

There is a faithful group action $\mathcal{D}_6 \rightarrow \text{Aut}(C)$, where C is a compact Riemann surface of genus $g(C) = 7$. A corresponding generating vector is given by $V_1 := (st, st, t^5, t^5)$. The stabilizer set Σ_1 of the action

$$\psi_1: G_1 \rightarrow G_1/K_1 \simeq \mathcal{D}_6 \rightarrow \text{Aut}(C)$$

fulfills the freeness conditions:

- i) $\Sigma_1 \cap \tau \Sigma_1 \tau^{-1} \cap \tau^2 \Sigma_1 \tau^{-2} = \{1_G\}$.
- ii) $(\tau g)^3 \notin \Sigma_1$ for all $g \in G^0$ and
- iii) $\tau f^2 \tau^{-1} \notin \Sigma_1$ for all $f \in G_1 \setminus G^0 \cap \Sigma_1$.

According to Proposition 3.5 c), the tuple $(G, G^0, K_1, \tau, h, V_1)$ is an algebraic datum of a threefold $X = C^3/G$ isogenous to a product. Since $g(C) = 7$, it holds

$$\chi(\mathcal{O}_X) = -\frac{(g(C) - 1)^3}{216} = -1.$$

For completeness, we also determine the Hodge numbers (cf. Section 4):

$$h^{3,0}(X) = 2, \quad h^{2,0}(X) = 1, \quad h^{1,0}(X) = 1, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 8.$$

Example 5.9.

Let $S = (C_1 \times C_2)/G$ be a rigid surface isogenous to a product of unmixed type, then $C_i/G \simeq \mathbb{P}^1$ and the G -covers $C_i \rightarrow \mathbb{P}^1$ are branched over $0, 1$ and ∞ . These surfaces are called Beauville surfaces, since Beauville provided the first example of such a surface (cf. [Be83]). In his example $G = \mathbb{Z}_5^2$ and $g(C_i) = 6$ yielding $\chi(\mathcal{O}_S) = 1$. Appropriate generating vectors V_i for \mathbb{Z}_5^2 are given by

$$V_1 = [(0, 3), (3, 3), (2, 4)] \quad \text{and} \quad V_2 = [(2, 0), (2, 1), (1, 4)].$$

We can easily modify this example to obtain a rigid threefold isogenous to a product with $\chi(\mathcal{O}_X) = -1$. Consider the generating vector $V_3 = (1, 1, 3)$ of \mathbb{Z}_5 . It corresponds to an action

$$\psi_3: \mathbb{Z}_5 \rightarrow \text{Aut}(C_3),$$

where C_3 is a curve of genus two, $C_3/\mathbb{Z}_5 \simeq \mathbb{P}^1$ and the \mathbb{Z}_5 -cover $C_3 \rightarrow \mathbb{P}^1$ is branched over $0, 1$ and ∞ . We obtain a diagonal, free action of \mathbb{Z}_5^2 on $C_1 \times C_2 \times C_3$, where \mathbb{Z}_5^2 acts on C_3 via ψ_3 composed with the projection to the first factor. The quotient

$$X = (C_1 \times C_2 \times C_3)/\mathbb{Z}_5^2$$

is a rigid threefold isogenous to a product with $\chi(\mathcal{O}_X) = -1$ and the Hodge numbers of X are the following:

$$h^{3,0}(X) = 3, \quad h^{2,0}(X) = 1, \quad h^{1,0}(X) = 0, \quad h^{1,1}(X) = 5 \quad \text{and} \quad h^{1,2}(X) = 9.$$

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Author's Address.

Christian Gleißner: Dipartimento di Matematica, Università degli Studi di Trento;
Via Sommarive 14; I-38123 Povo (Trento), Italy